# Entanglement and its Verification

## NYC Quantum Meetup
## 9/19/2017

OR HOW I LEARNED TO STOP WORRYING AND LOVE NON-LOCAL THEORIES

MUIR KUMPH

IBM RESEARCH 2017

# Entanglement is the 'Magic' of Quantum Mechanics

- there is no end of weird and 'new age' descriptions
  - all particles in the universe are connected
  - spooky action at a distance (Einstein)
- seems to be needed for exponential speed-up
- today we do it with math
- for a thorough review – see Horodecki et. al

REVIEWS OF MODERN PHYSICS, VOLUME 81, APRIL–JUNE 2009

**Quantum entanglement**

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(Published 17 June 2009)

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

**Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?**

A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

# Entanglement Verification is Important

- entanglement is necessary (but not sufficient) for an exponential speed-up
- otherwise, the machine cannot be what I would call a 'Quantum Computer'
- eg. D-wave

PHYSICAL REVIEW X 4, 021041 (2014)

**Entanglement in a Quantum Annealing Processor**

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- **at best** it looks like D-Wave built an 8 qubit quantum annealer
- thousand of qubits are just bits if no entanglement can be demonstrated
- when companies claim large numbers of qubits, check to see if they are entangled

# What does it mean?

- entangled states can have the property of *non-locality*
- measuring one part of a system of entangled particles has an effect on the other particles
- 3 qubit system is the simplest system to demonstrate this without resorting to statistics – just need a few measurements

# Outline Today

- all the quantum theory necessary to understand 3 qubit entanglement
- introduction to probability
- 2 qubit Bell/CHSH Inequalities

# Fundamental Quantum State - Qubit

- when measured – it will be projected into one of two states $|0\rangle$ or $|1\rangle$
- can be in superposition of its states: $\alpha|0\rangle + \beta|1\rangle$
- when the qubit is in a superposition and it is measured
  - probability p(0) that its $|0\rangle$ is $|\alpha|^2$
  - probability p(1) that its $|1\rangle$ is $|\beta|^2$
- the state must be normalized so that $|\alpha|^2 + |\beta|^2 = 1$

# bras and kets

- kets
  - $|0\rangle = \binom{1}{0}$
  - $|1\rangle = \binom{0}{1}$
- bras
  - $\langle 0| = (1 \quad 0)$
  - $\langle 1| = (0 \quad 1)$
- brakets
  - $\langle 0|0\rangle = 1$
  - $\langle 1|1\rangle = 1$
  - $\langle 0|1\rangle = 0$
  - $\langle 1|0\rangle = 0$

# Multiqubit States

- any quantum state can be written as a *superposition* of *product* states
- product states are essentially just lists of the states of each individual system – also called a *tensor product*
  - $|010\rangle$ is a product of 3 different qubits with the states $|0\rangle$ $|1\rangle$ and $|0\rangle$
  - the ordering is implicit
- superposition of these product states can give rise to entanglement
  - eg. $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

# Entanglement

- definition: a state which cannot be written as a product state
- $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$ is not an entangled state
  - because its $|0+\rangle$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
- $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is entangled

# Expectation Values

- measurement of qubits is probabilistic
  - a useful quantity is the average (or expected value) of an infinite series of measurements – or ensemble of identical quantum states (see Ergodic theorem)

- define a measurement operator $\mathcal{Z}_1 = |0\rangle\langle 0| - |1\rangle\langle 1|$
  - linear algebra: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- expectation of this operator acting on $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is $\langle\psi|\, \mathcal{Z}_1 |\psi\rangle$
  - gives the value $\alpha^2 - \beta^2$
  - $\langle 0|\, \mathcal{Z}_1 |0\rangle = 1$ or with linear algebra: $(1 \quad 0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
  - $\langle 1|\, \mathcal{Z}_1 |1\rangle = -1$

# Pop Quiz

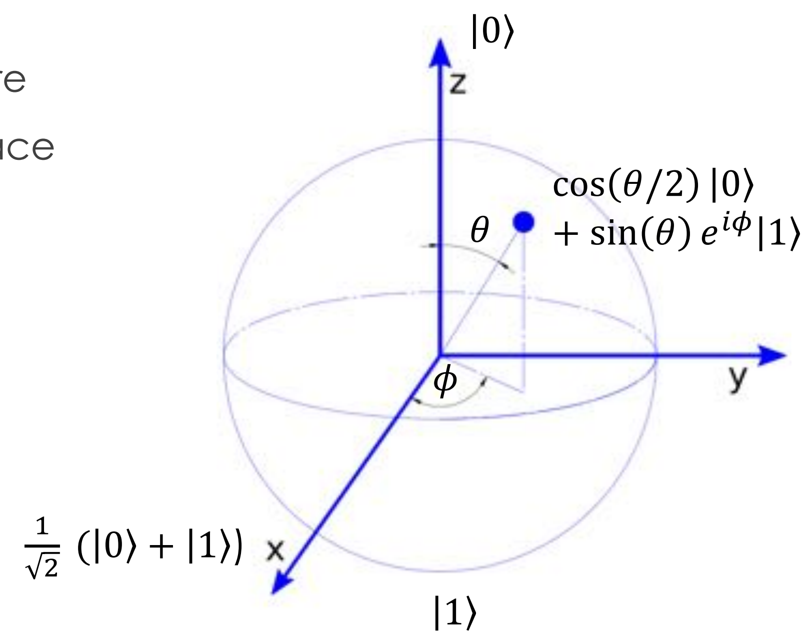
- $\mathcal{Z}_1 = |0\rangle\langle 0| - |1\rangle\langle 1|$
- $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

- what is $\langle +| \mathcal{Z}_1 |+\rangle$?
- remember to keep order of bras and kets!
- $\langle +|(|0\rangle\langle 0| - |1\rangle\langle 1|)|+\rangle$
- $\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)(|0\rangle\langle 0| - |1\rangle\langle 1|)\ \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

# Bloch Sphere

- qubit state can be depicted on Bloch sphere
- state is represented by the point at the surface
- $|0\rangle$ state is pointing straight up
  - its projection along the z axis is equal to 1
- $|1\rangle$ state is pointing straight down
  - its projection along the z axis is equal to -1

- $|\psi\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2)\,e^{i\phi}|1\rangle$



# 3 Qubit System

- GHZ (Greenberger-Horne-Zeilinger) state: $\frac{|000\rangle - |111\rangle}{\sqrt{2}}$

  - Or if you prefer linear algebra: $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

# Expectation of Single Qubit in GHZ

- To avoid measuring a qubit, use the identity operator $\mathcal{I}_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$
- expectation value of first qubit is: $\frac{\langle 000| - \langle 111|}{\sqrt{2}} \mathcal{Z}_1 \mathcal{I}_2 \mathcal{I}_3 \frac{|000\rangle - |111\rangle}{\sqrt{2}}$=0
- $\mathcal{Z}_1 \mathcal{I}_2 \mathcal{I}_3 = |000\rangle\langle 000| + |001\rangle\langle 001| + |010\rangle\langle 010| + |011\rangle\langle 011| - |100\rangle\langle 100| - |101\rangle\langle 101| - |110\rangle\langle 110| - |111\rangle\langle 111|$
- linear algebra: $\left(\frac{1}{\sqrt{2}} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -\frac{1}{\sqrt{2}}\right) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

# Pop Quiz

- compute the expectation value of first qubit: $\frac{\langle 000| - \langle 111|}{\sqrt{2}} \mathcal{Z}_1 \mathcal{I}_2 \mathcal{I}_3 \frac{|000\rangle - |111\rangle}{\sqrt{2}}$=0
- $\mathcal{Z}_1 \mathcal{I}_2 \mathcal{I}_3 = |000\rangle\langle 000| + |001\rangle\langle 001| + |010\rangle\langle 010| + |011\rangle\langle 011| - |100\rangle\langle 100| - |101\rangle\langle 101| - |110\rangle\langle 110| - |111\rangle\langle 111|$

# Expectation Values of Parity Measurement

- Multiply operators together: $\mathcal{Z}_1\mathcal{Z}_2\mathcal{Z}_3$
  - linear algebra (Tensor Product) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

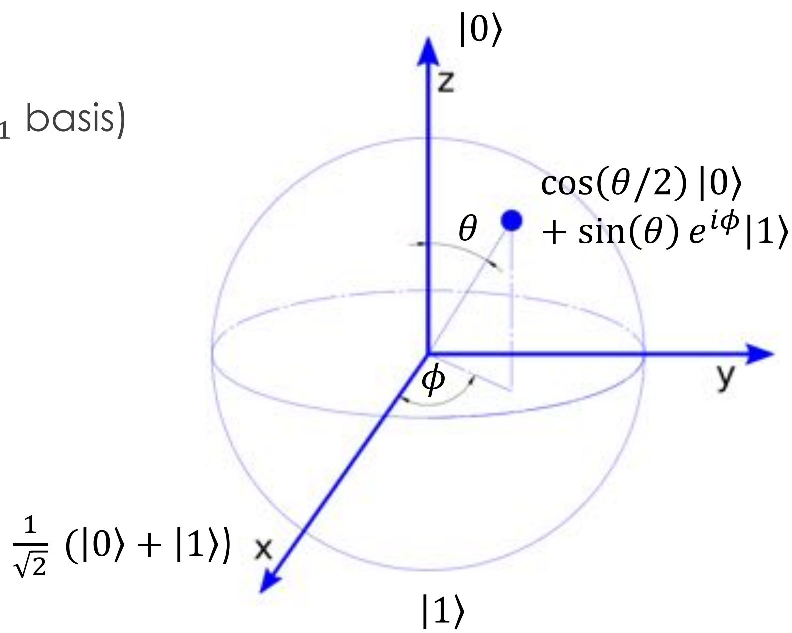
# Parity or Combined Expectations

- measuring the parity of the GHZ state in the Z basis now simplifies to:
  - $\frac{1}{2}(\langle 000|\mathcal{Z}_1\mathcal{Z}_2\mathcal{Z}_3|000\rangle - \langle 111|\mathcal{Z}_1\mathcal{Z}_2\mathcal{Z}_3|000\rangle - \langle 000|\mathcal{Z}_1\mathcal{Z}_2\mathcal{Z}_3|111\rangle + \langle 111|\mathcal{Z}_1\mathcal{Z}_2\mathcal{Z}_3|111\rangle)$

- linear algebra notation:
  - $(1/2)(1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1)\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

# Measuring in Other Basis

- $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
- we can measure in the + basis (also called the $\mathcal{X}_1$ basis)
- $\mathcal{X} \quad = |1\rangle\langle 0| + |0\rangle\langle 1|$
- $\mathcal{Y} \quad = i|1\rangle\langle 0| - i|0\rangle\langle 1|$



# Parity Measurements of the GHZ State

- consider the following measurements on a GHZ state
  - $\mathcal{X}_1\,\mathcal{Y}_2\,\mathcal{Y}_3$
  - $\mathcal{Y}_1\,\mathcal{X}_2\,\mathcal{Y}_3$
  - $\mathcal{Y}_1\,\mathcal{Y}_2\,\mathcal{X}_3$

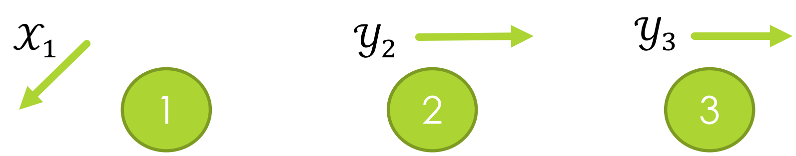
- for instance:
  - $\mathcal{X}_1\,\mathcal{Y}_2\,\mathcal{Y}_3 = -|111\rangle\langle 000| + |110\rangle\langle 001| + |101\rangle\langle 010| - |100\rangle\langle 011| - |011\rangle\langle 100| + |010\rangle\langle 101| + |001\rangle\langle 110| - |000\rangle\langle 111|$
  - $\frac{\langle 000| - \langle 111|}{\sqrt{2}}\mathcal{X}_1\,\mathcal{Y}_2\,\mathcal{Y}_3\,\frac{|000\rangle - |111\rangle}{\sqrt{2}}$=1

# Local Hidden Variable Theories

- imagine there are three people who are given different colored balls, say red green and blue
- without looking at the balls are put into their pockets and they walk far away from each other
- there is an unknown assignment of the colored balls and the people (reality)
- if you ask one person to take out a ball and tell you the color, then the colors of the other balls don't change (local)
- measuring in different basis could be thought of as asking different questions about the balls – like how big is it, or what is it made of?

# Locality and Reality of Multi-qubit Systems

- we are about to disprove this! **Proof by contradiction**
- there should be some sort of hidden value which is revealed by measurement (reality)

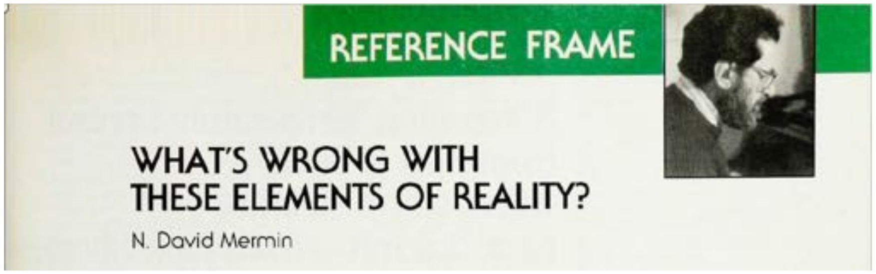



- the measurement of say qubit 1 should not change the hidden value of another qubit (locality)

# Local and Real Qubits

- we should be able to replace the qubit operators with the hidden (real) value of the measurement
  - $m_x^1 m_y^2 m_y^3$
  - $m_y^1 m_x^2 m_y^3$
  - $m_y^1 m_y^2 m_x^3$
- if these measurements were performed on locally independent qubits, then we could vary the measurements on the other qubits without modifying the result of the one in question.

Mermin, N. D. (1990). What's wrong with these elements of reality? *Physics Today*, *43*(6), 9–11. https://doi.org/10.1063/1.2810588



# Parity Measurements of Local Qubits

- each measurement $m_x^i$, $m_y^i$ will return either -1 or 1,
  - $m_x^1 m_y^2 m_y^3 = 1$ (from quantum theory)
  - $m_y^1 m_x^2 m_y^3 = 1$ (from quantum theory)
  - $m_y^1 m_y^2 m_x^3 = 1$ (from quantum theory)
  - ------------
  - $m_x^1 m_x^2 m_x^3$ should then equal 1 (from local realism)
- multiply through the columns to get the bottom row
  - $\left(m_y^i\right)^2 = 1$

# Parity Measurements of Local Qubits

- But measurements of $\mathcal{X}_1\mathcal{X}_2\mathcal{X}_3$ don't return 1
- $\mathcal{X}_1\mathcal{X}_2\mathcal{X}_3 = |111\rangle\langle 000| + |110\rangle\langle 001| + |101\rangle\langle 010| + |100\rangle\langle 011| + |011\rangle\langle 100| + |010\rangle\langle 101| + |001\rangle\langle 110| + |000\rangle\langle 111|$
- $\frac{\langle 000| - \langle 111|}{\sqrt{2}} \mathcal{X}_1\mathcal{X}_2\mathcal{X}_3 \frac{|000\rangle - |111\rangle}{\sqrt{2}}$ = -1

# Pop Quiz

- compute the expected value of the $\mathcal{X}_1\mathcal{X}_2\mathcal{X}_3$ measurement on the GHZ state
- $\mathcal{X}_1\mathcal{X}_2\mathcal{X}_3 = |111\rangle\langle 000| + |110\rangle\langle 001| + |101\rangle\langle 010| + |100\rangle\langle 011| + |011\rangle\langle 100| + |010\rangle\langle 101| + |001\rangle\langle 110| + |000\rangle\langle 111|$
- $\frac{|000\rangle - |111\rangle}{\sqrt{2}}$

# Reality or Locality?

- quantum theory of a GHZ state along with the assumptions of reality and locality lead to a contradiction
- quantum theory: seems to be right
- reality: a cherished concept – that there is one common reality for us all – independent of observation made
- locality: has to go
  - measurements on one part of a quantum system can change the other parts of the system.

# what about less than 3 qubits?

- GHZ states can be hard to create
- 2 qubits are sufficient to demonstrate entanglement
- Bell first notice this in 1969
- Clauser, Horne, Shimony and Holt

Physics Vol. 1, No. 3, pp. 195–200, 1964 Physics Publishing Co. Printed in the United States

ON THE EINSTEIN PODOLSKY ROSEN PARADOX*

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*(Received 4 November 1964)*

PROPOSED EXPERIMENT TO TEST LOCAL HIDDEN-VARIABLE THEORIES*

John F. Clauser†
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and

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and

Abner Shimony
Departments of Philosophy and Physics, Boston University, Boston, Massachusetts 02215

and

Richard A. Holt
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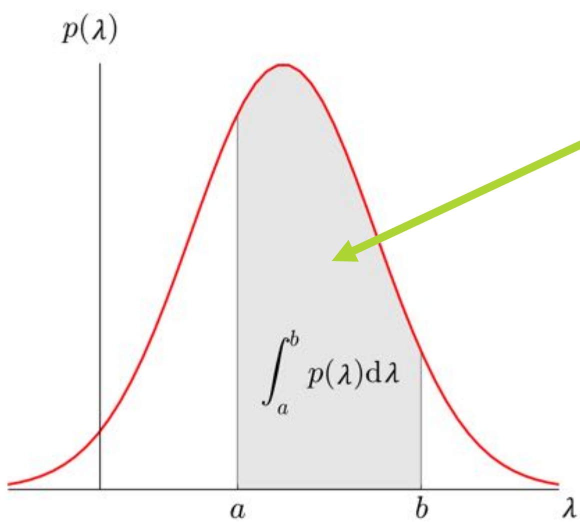
(Received 4 August 1969)

# Bell/CHSH Inequalities

- can show a violation of local realism with just two qubits using statistics of the joint expectation value
- in classical statistics, the joint expected value of the product of two measurements A and B is
  - $E(a,b) = \int \underline{A}(a,\lambda)\underline{B}(b,\lambda)\rho(\lambda)\,d\lambda$
  - $a$ and $b$ are settings for the measurements (like the measurement basis angle)
  - $\underline{A}(a,\lambda)\,\underline{B}(b,\lambda)$ are the average values of the measurements depending upon the settings $a$, $b$, and a hidden unknown variable $\lambda$
  - $\rho(\lambda)$ is the probability distribution of the unknown variable $\lambda$

# Probability Distribution Functions (PDF)

- The probability of all possible values of the variable $\lambda$ must be 1
  - $\int \rho(\lambda)\, d\lambda = 1$




probability of finding $\lambda$ between a and b

# Joint Expectation Value

- $E(a,b) = \int \underline{A}(a,\lambda)\underline{B}(b,\lambda)\rho(\lambda)\, d\lambda$
- these are the results of measurements on the qubits so, $|\underline{A}| \leq 1$, $|\underline{B}| \leq 1$
- consider the difference between two expectation values with different settings:

- $E(a,b) - E(a,b') = \int \underline{A}(a,\lambda)\underline{B}(b,\lambda)\rho(\lambda)\, d\lambda$ - $\int \underline{A}(a,\lambda)\underline{B}(b',\lambda)\rho(\lambda)\, d\lambda$
- $= \int [\ \underline{A}(a,\lambda)\underline{B}(b,\lambda) - \underline{A}(a,\lambda)\underline{B}(b',\lambda)]\ \rho(\lambda)\, d\lambda$
- $= \int \underline{A}(a,\lambda)\underline{B}(b,\lambda)[1 \pm \underline{A}(a',\lambda)\underline{B}(b',\lambda)]\rho(\lambda)\, d\lambda - \int \underline{A}(a,\lambda)\underline{B}(b',\lambda)[1 \pm \underline{A}(a',\lambda)\underline{B}(b,\lambda)]\rho(\lambda)\, d\lambda$

# Joint Expectation Value

- for any numbers X, Y, Z: $X = Y - Z \rightarrow |X| \leq |Y| + |Z|$
- $E(a,b) - E(a,b') = \int \underline{A}(a,\lambda)\underline{B}(b,\lambda)[1 \pm \underline{A}(a',\lambda)\underline{B}(b',\lambda)]\rho(\lambda)\, d\lambda - \int \underline{A}(a,\lambda)\underline{B}(b',\lambda)[1 \pm \underline{A}(a',\lambda)\underline{B}(b,\lambda)]\rho(\lambda)\, d\lambda$
- $|E(a,b) - E(a,b')| \leq |\int \underline{A}(a,\lambda)\underline{B}(b,\lambda)\left[1 \pm \underline{A}(a',\lambda)\underline{B}(b',\lambda)\right]\rho(\lambda)\, d\lambda| + |\int \underline{A}(a,\lambda)\underline{B}(b',\lambda)\left[1 \pm \underline{A}(a',\lambda)\underline{B}(b,\lambda)\right]\rho(\lambda)\, d\lambda|$
- $[1 \pm \underline{A}(a',\lambda)\underline{B}(b',\lambda)]\rho(\lambda)$ and $[1 \pm \underline{A}(a',\lambda)\underline{B}(b,\lambda)]\rho(\lambda)$ are both positive
- $\leq \int\left|\underline{A}(a,\lambda)\underline{B}(b,\lambda)\right| \left|\left[1 \pm \underline{A}(a',\lambda)\underline{B}(b',\lambda)\right]\rho(\lambda)\, d\lambda\right| + \int\left|\underline{A}(a,\lambda)\underline{B}(b',\lambda)\right| \left|\left[1 \pm \underline{A}(a',\lambda)\underline{B}(b,\lambda)\right]\rho(\lambda)\, d\lambda\right|$
- $|\underline{A}| \leq 1,\ |\underline{B}| \leq 1$
- $|E(a,b) - E(a,b')| \leq \int \left|\left[1 \pm \underline{A}(a',\lambda)\underline{B}(b',\lambda)\right]\rho(\lambda)\, d\lambda\right| + \int \left|\left[1 \pm \underline{A}(a',\lambda)\underline{B}(b,\lambda)\right]\rho(\lambda)\, d\lambda\right|$

# CHSH inequality

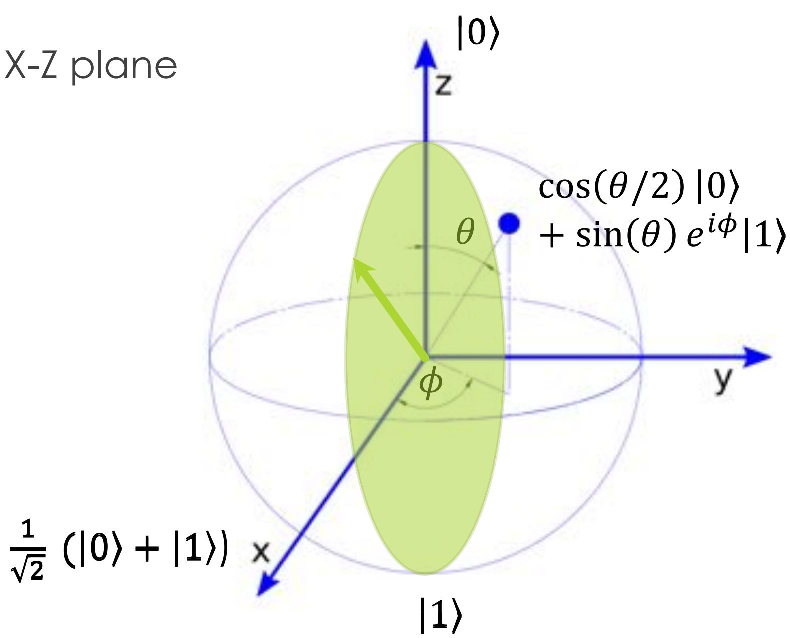
- $|E(a,b) - E(a,b')| \leq \int [1 \pm \underline{A}(a',\lambda)\underline{B}(b',\lambda)]\rho(\lambda)\, d\lambda + \int [1 \pm \underline{A}(a',\lambda)\underline{B}(b,\lambda)]\rho(\lambda)\, d\lambda$
- $\int \rho(\lambda)\, d\lambda = 1$
- $|E(a,b) - E(a,b')| \leq 2 \pm \int [\underline{A}(a',\lambda)\underline{B}(b',\lambda)]\rho(\lambda)\, d\lambda + \int [\underline{A}(a',\lambda)\underline{B}(b,\lambda)]\rho(\lambda)\, d\lambda$
- $|E(a,b) - E(a,b')| \leq 2 \pm [E(a',b') + E(a',b)]$
- $|E(a,b) - E(a,b')| \leq 2 - |E(a',b') + E(a',b)|$
- $2 \geq |E(a,b) - E(a,b')| + |E(a',b') + E(a',b)|$
- $2 \geq |E(a,b) - E(a,b')| + |E(a',b') + E(a',b)|$
- $2 \geq |E(a,b) - E(a,b') + E(a',b') + E(a',b)|$

# Checking the Bell Inequality

- Back to Quantum Theory
- $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is entangled
- we can check to see if it violates the Bell inequality
- first we need to introduce a new type of measurement
- measurement along other directions than just $\mathcal{X}_1\ \mathcal{Y}_1\ \mathcal{Z}_1$

# Measurement along Vector

- measurement along a direction in the X-Z plane
- $\mathcal{Z}_1 = |0\rangle\langle 0| - |1\rangle\langle 1|$
- $\mathcal{X}_1 = |1\rangle\langle 0| + |0\rangle\langle 1|$
- $A_1(a) = \sin(a)\, \mathcal{X}_1 + \cos(a)\, \mathcal{Z}_1$



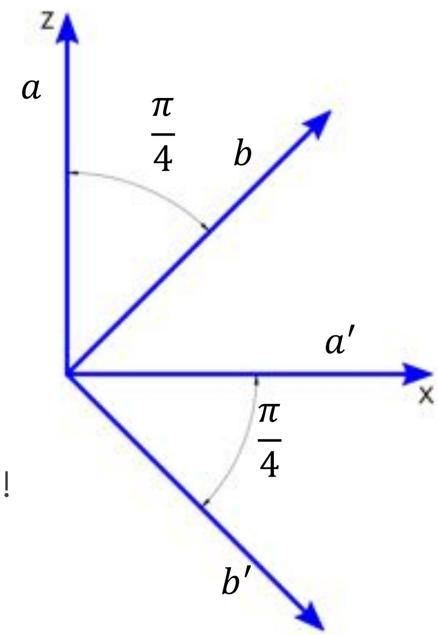
# Expected Value Along Vectors

- $A_1(a)B_2(b) = \sin(a)\sin(b)|11\rangle\langle 00| + \sin(a)\sin(b)|10\rangle\langle 01| + \sin(a)\sin(b)|01\rangle\langle 10| + \sin(a)\sin(b)|00\rangle\langle 11| + \cos(a)\cos(b)|00\rangle\langle 00| - \cos(a)\cos(b)|01\rangle\langle 01| - \cos(a)\cos(b)|10\rangle\langle 10| + \cos(a)\cos(b)|11\rangle\langle 11| + \cos(a)\sin(b)|01\rangle\langle 00| + \sin(a)\cos(b)|10\rangle\langle 00| + \cos(a)\sin(b)|00\rangle\langle 01| - \sin(a)\cos(b)|11\rangle\langle 01| + \sin(a)\cos(b)|00\rangle\langle 10| - \cos(a)\sin(b)|11\rangle\langle 10| - \sin(a)\cos(b)|01\rangle\langle 11| - \cos(a)\sin(b)|10\rangle\langle 11|$
- $\left(\frac{\langle 00|}{\sqrt{2}} + \frac{\langle 11|}{\sqrt{2}}\right) A_1(a)B_2(b) \left(\frac{|00\rangle}{\sqrt{2}} + \frac{|11\rangle}{\sqrt{2}}\right) = \sin(a)\sin(b) + \cos(a)\cos(b) = \cos(a-b)$

# Bell-CHSH quantum mechanics

- $a = \frac{\pi}{2},\ a' = 0,\ \ b = \frac{\pi}{4},\ \ b' = -\frac{\pi}{4}$
- $E(a,b) = -E(a,b') = E(a',b) = E(a',b') = \frac{1}{\sqrt{2}}$
- $S = E(a,b) - E(a,b') + E(a',b) + E(a',b') = 2\sqrt{2}$

- but from local realism
  - $2 \geq |E(a,b) - E(a,b') + E(a',b') + E(a',b)|$
  - since the entanglement witness S is larger than 2, its not locally real!



# What we learned

- Entanglement can be verified with Bell states using statistics
- Entanglement can be verified with GHZ states – single shot measurements

- All the quantum theory to describe multi-qubit states
- Multi-qubit parity measurements – the basis for quantum error correction

# Next Time

- Explore algorithms which use entanglement for exponential speedup
  - Deutsch-Josza
  - Shor
  - Quantum Chemistry
  - Quantum Algorithm for Linear Systems of Equations (Harrow, Hassidim, and Lloyd)

# IBM

End of Entanglement and its Verification