Dirac notation

- Just another way of describing vectors:

$$\mathbf{v} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{bmatrix} = |\mathbf{v}\rangle$$

- and their duals:

$$\langle\mathbf{v}| = \overline{\mathbf{v}^{\mathrm{T}}} = \begin{bmatrix} \overline{v_0} & \overline{v_1} & \dots & \overline{v_n} \end{bmatrix}$$

- Convenient for describing vectors in the Hilbert space $\mathbb{C}^n$, the vector space of quantum mechanics

$\mathbb{C}^n$ and the inner product

- A Hilbert space, for our (finite) purposes, is a vector space with an *inner product*, and a *norm* defined by that inner product. We use the following in $\mathbb{C}^n$:
  - The inner product assigns a scalar value to each pair of vectors:

$$\langle \mathbf{u}|\mathbf{v}\rangle = \overline{\mathbf{u}^{\mathrm{T}}}\mathbf{v} = \begin{bmatrix}\overline{u_0} & \overline{u_1} & \dots & \overline{u_n}\end{bmatrix}\begin{bmatrix}v_0 \\ v_1 \\ \vdots \\ v_n\end{bmatrix} = \overline{u_0}\cdot v_0 + \overline{u_1}\cdot v_1 + \ldots + \overline{u_n}\cdot v_n$$

  - The norm is the square root of the inner product of a vector with itself (i.e. Euclidean norm, ℓ^2-norm, 2-norm over complex numbers):

$$\||\mathbf{v}\rangle\| = \sqrt{\langle \mathbf{v}|\mathbf{v}\rangle}$$

  - Geometrically, this norm gives the distance from the origin to the point $|\mathbf{v}\rangle$ that follows from the Pythagorean theorem.

Properties of the inner product

The inner product satisfies the three following properties:

Definition

1. $\langle \mathbf{v}|\mathbf{v}\rangle \geq 0$, with $\langle \mathbf{v}|\mathbf{v}\rangle = 0$ if and only if $|\mathbf{v}\rangle = \mathbf{0}$.
2. $\langle \mathbf{u}|\mathbf{v}\rangle = \overline{\langle \mathbf{v}|\mathbf{u}\rangle}$ for all $|\mathbf{u}\rangle$, $|\mathbf{v}\rangle$ in the vector space.
3. $\langle \mathbf{u}|\alpha_0\mathbf{v} + \alpha_1\mathbf{w}\rangle = \alpha_0 \langle \mathbf{u}|\mathbf{v}\rangle + \alpha_1 \langle \mathbf{u}|\mathbf{w}\rangle$.
   More generally, the inner product of $|\mathbf{u}\rangle$ and $\sum_i \alpha_i |\mathbf{v}_i\rangle$ is equal to $\sum_i \alpha_i \langle \mathbf{u}|\mathbf{v}_i\rangle$ for all scalars α_i and vectors $|\mathbf{u}\rangle$, $|\mathbf{v}\rangle$ in the vector space (this is known as *linearity in the second argument*).

The outer product

- The *outer product* is the *tensor* or *Kronecker product* of a vector with the conjugate transpose of another. The result is not a scalar, but a matrix:

$$|\mathbf{v}\rangle \langle\mathbf{u}| = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} \overline{u_0} & \overline{u_1} & \dots & \overline{u_m} \end{bmatrix} = \begin{bmatrix} v_0\overline{u_0} & v_0\overline{u_1} & \dots & v_0\overline{u_m} \\ v_1\overline{u_0} & v_1\overline{u_1} & \dots & v_1\overline{u_m} \\ \vdots & \vdots & \ddots & \vdots \\ v_n\overline{u_0} & v_n\overline{u_1} & \dots & v_n\overline{u_m} \end{bmatrix}$$

- Often used to describe a linear transformation between vector spaces.
- A linear transformation from a Hilbert space U to another Hilbert space V on a vector $|\mathbf{w}\rangle$ in U may be succintly described in Dirac notation:

$$(|\mathbf{v}\rangle \langle\mathbf{u}|)\,|\mathbf{w}\rangle = |\mathbf{v}\rangle \langle\mathbf{u}|\mathbf{w}\rangle = \langle\mathbf{u}|\mathbf{w}\rangle\,|\mathbf{v}\rangle$$

  Since $\langle\mathbf{u}|\mathbf{w}\rangle$ is a commutative, scalar value.

The tensor product

- Usually simplified from $|\mathbf{u}\rangle \otimes |\mathbf{v}\rangle$ to $|\mathbf{u}\rangle\,|\mathbf{v}\rangle$ or $|\mathbf{uv}\rangle$
- A vector tensored with itself n times is denoted $|\mathbf{v}\rangle^{\otimes n}$ or $|\mathbf{v}\rangle^{n}$
- Two column vectors $|\mathbf{u}\rangle$ and $|\mathbf{v}\rangle$ of lengths m and n yield a column vector of length $m \cdot n$ when tensored:

$$|\mathbf{u}\rangle\,|\mathbf{v}\rangle = |\mathbf{uv}\rangle = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_m \end{bmatrix} \otimes \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_0 \cdot v_0 \\ u_0 \cdot v_1 \\ \vdots \\ u_0 \cdot v_n \\ u_1 \cdot v_0 \\ \vdots \\ u_{m-1} \cdot v_n \\ u_m \cdot v_0 \\ \vdots \\ u_m \cdot v_n \end{bmatrix}$$

$\mathbb{C}^2$ describes a single quantum bit (qubit)

- A classical bit may be represented as a base-2 number that takes either the value 1 or the value 0
- Qubits are also base-2 numbers, but in a superposition of the measurable values 1 and 0
- The state of a qubit at any given time represented as a two-dimensional *state space* in $\mathbb{C}^2$ with *orthonormal* basis vectors $|1\rangle$ and $|0\rangle$
- The superposition $|\psi\rangle$ of a qubit is represented as a linear combination of those basis vectors:

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle$$

Where a_0 is the complex scalar *amplitude* of measuring $|0\rangle$, and a_1 the amplitude of measuring the value $|1\rangle$.

Amplitudes, not probabilities

- Amplitudes may be thought of as "quantum probabilities" in that they represent the chance that a given quantum state will be observed when the superposition is collapsed
- Most fundamental difference between probabilities of states in classical probabilistic algorithms and amplitudes: amplitudes are complex
  - Complex numbers required to fully describe superposition of states, interference or entanglement in quantum systems.[1]
  - As the probabilities of a classical system must sum to 1, so too the squares of the absolute values of the amplitudes of states in a quantum system must add up to 1

[1] See `http://www.scottaaronson.com/democritus/lec9.html` for a great discussion by of why complex numbers and the 2-norm are used to describe quantum mechanical systems

Amplitudes and the normalization condition

- Just as the hardware underlying the bits of a classical computer may vary in voltage, quantum systems are not usually so perfectly behaved
- An assumption is made about quantum state vectors called the *normalization conditon*: $|\psi\rangle$ is a unit vector.
  - $\| |\psi\rangle \| = \langle\psi|\psi\rangle = 1$
  - If $|0\rangle$ and $|1\rangle$ are orthonormal, then by orthogonality $\langle 0|1\rangle = \langle 1|0\rangle = 0$, and by normality $\langle 0|0\rangle = \langle 1|1\rangle = 1$
  - It follows that $|a_0|^2 + |a_1|^2 = 1$:

$$\begin{aligned} 1 &= \langle\psi|\psi\rangle \\ &= (\overline{a_0}\langle 0| + \overline{a_1}\langle 1|) \cdot (a_0|0\rangle + a_1|1\rangle) \\ &= |a_0|^2\langle 0|0\rangle + |a_1|^2\langle 1|1\rangle + \overline{a_1}a_0\langle 1|0\rangle + \overline{a_0}a_1\langle 0|1\rangle \\ &= |a_0|^2 + |a_1|^2 \end{aligned}$$

Why we use Dirac notation

The following is equivalent to the last slide:

$$
\begin{aligned}
1 &= \langle\psi|\psi\rangle \\
&= \left(\overline{a_0}\langle 0| + \overline{a_1}\langle 1|\right)\cdot\left(a_0|0\rangle + a_1|1\rangle\right) \\
&= \left(\overline{a_0}\begin{bmatrix}\overline{\psi_{00}} & \overline{\psi_{01}}\end{bmatrix} + \overline{a_1}\begin{bmatrix}\overline{\psi_{10}} & \overline{\psi_{11}}\end{bmatrix}\right)\cdot\left(a_0\begin{bmatrix}\psi_{00}\\ \psi_{01}\end{bmatrix} + a_1\begin{bmatrix}\psi_{10}\\ \psi_{11}\end{bmatrix}\right) \\
&= \begin{bmatrix}\overline{a_0\psi_{00}} + \overline{a_1\psi_{10}} & \overline{a_0\psi_{01}} + \overline{a_1\psi_{11}}\end{bmatrix}\cdot\begin{bmatrix}a_0\psi_{00} + a_1\psi_{10}\\ a_0\psi_{01} + a_1\psi_{11}\end{bmatrix} \\
&= \overline{a_0\psi_{00}}a_0\psi_{00} + \overline{a_1\psi_{10}}a_0\psi_{00} + \overline{a_0\psi_{00}}a_1\psi_{10} + \overline{a_1\psi_{10}}a_1\psi_{10} \\
&\quad + \overline{a_0\psi_{01}}a_0\psi_{01} + \overline{a_1\psi_{11}}a_0\psi_{01} + \overline{a_0\psi_{01}}a_1\psi_{11} + \overline{a_1\psi_{11}}a_1\psi_{11} \\
&= |a_0|^2\left(|\psi_{00}|^2 + |\psi_{01}|^2\right) + |a_1|^2\left(|\psi_{10}|^2 + |\psi_{11}|^2\right) \\
&\quad + \overline{a_1}a_0\left(\overline{\psi_{10}}\psi_{00} + \overline{\psi_{11}}\psi_{01}\right) + \overline{a_0}a_1\left(\overline{\psi_{00}}\psi_{10} + \overline{\psi_{01}}\psi_{11}\right) \\
&= |a_0|^2 + |a_1|^2
\end{aligned}
$$

The computational basis

- $|0\rangle$ and $|1\rangle$ may be transformed into any two vectors that form an orthonormal basis in $\mathbb{C}^2$
- The most common basis used in quantum computing is called the *computational basis*:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- The computational basis tends to be the most straightforward basis for computing and understanding quantum algorithms
- Assume I'm using the computational basis unless otherwise stated

Another basis

- Any other orthonormal basis could be used:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

- Providing a slightly different but equivalent way of expressing of a qubit:

$$\begin{aligned}|\psi\rangle &= a_0 |0\rangle + a_1 |1\rangle \\ &= a_0 \frac{|+\rangle + |-\rangle}{\sqrt{2}} + a_1 \frac{|+\rangle - |-\rangle}{\sqrt{2}} \\ &= \frac{a_0 + a_1}{\sqrt{2}} |+\rangle + \frac{a_0 + a_1}{\sqrt{2}} |-\rangle\end{aligned}$$

- Here, instead of measuring the states $|0\rangle$ and $|1\rangle$ each with respective probabilities $|a_0|^2$ and $|a_1|^2$, the states $|+\rangle$ and $|-\rangle$ would be measured with probabilities $|a_0 + a_1|^2/2$ and $|a_0 - a_1|^2/2$.

Registers more useful than single qubits

- Each qubit in a quantum register is in a superposition of $|1\rangle$ and $|0\rangle$
- Consequently, a register of n qubits is in a superposition of all 2^n possible bit strings that could be represented using n bits
- The state space of a size-n quantum register is a linear combination of n basis vectors, each of length 2^n:

$$|\psi_n\rangle = \sum_{i=0}^{2^n-1} a_i |i\rangle$$

- A three-qubit register would thus have the following expansion:

$$\begin{aligned} |\psi_2\rangle = a_0 |000\rangle + a_1 |001\rangle + a_2 |010\rangle + a_3 |011\rangle \\ + a_4 |100\rangle + a_5 |101\rangle + a_6 |110\rangle + a_7 |111\rangle \end{aligned}$$